Euler Coefficients and Restricted Dyck Paths

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We consider the problem of enumerating Dyck paths staying weakly above the x-axis with a limit to the number of consecutive $\searrow$ steps, or a limit to the number of consecutive $\nearrow$ steps. We use Finite Operator Calculus to obtain formulas for the number of all such paths reaching a given point in the first quadrant. All our results are based on the Eulerian coefficients.

1 Introduction

One of the most recent papers on patterns occurring k times in Dyck paths was written by A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, 2007, to appear in *Discrete Mathematics* [5]. The authors find generating functions for all 16 patterns generated by combinations of four up ($\nearrow$) and down ($\searrow$) steps. A Dyck path starts at $(0,0)$, takes only up and down steps, and ends at $(2n,0)$, staying weakly above the x-axis. Returning to the x-axis at the end of the path has the advantage that every path containing the pattern $uduu$, say, k times, will contain the reversed pattern $ddud$ also k times when read backwards. This reduces significantly the number of patterns under consideration. Dyck paths containing k strings of length 3 were discussed by E. Deutsch in [1].

In this paper we consider only the patterns u^r and d^r, for all integers $r > 2$, and we will investigate only the case $k=0$, which means pattern avoidance. It has been shown in [5] that the generating function $f(t)$ for avoiding u^r(or d^r) satisfies the equation $f(t) = 1+\sum_{i=1}^{r-1} t^i f(t)^i = \frac{1-t-t^r f(t)^r}{1-2t}$. However, we will allow the Dyck paths to end at (n,m), $m \geq 0$, which removes the above mentioned symmetry, as shown in the following two tables.

m														
7												1		10
6											3		19	
5								1		6		28		112
4							2		9		33		116	
3				1		3		10		32		101		321
2			1		3		8		23		68		205	
1		1		2		5		13		36		104		309
0	1		1		2		5		13		36		104	
n :	0	1	2	3	4	5	6	7	8	9	10	11	12	13

The number of Dyck paths avoiding $uuuu$

m														
7								1		8		44		208
6							1		7		35		154	
5						1		6		27		110		423
4					1		5		20		75		270	
3				1		4		14		48		161		536
2			1		3		9		28		87		273	
1		1		2		5		14		40		118		357
0	1		1		2		5		13		36		104	
$n:$	0	1	2	3	4	5	6	7	8	9	10	11	12	13

The number of Dyck paths avoiding *dddd*

The two tables indicate the differences between the two problems, both starting out from equal counts on the x-axis ($m = 0$). Because only points (n, m) with $n + m = 0 \mod 2$ can be reached by a Dyck path, we consider the lattice points $(2n + b, 2m + b)$, for $b = 0, 1$. We first show that the number of Dyck paths to $(2n + b, 2m + b)$ avoiding d^r equals

$$Dyck\,(2n + b, 2m + b; d^r) = \frac{2m + b + 1}{n + m + b + 1} \binom{n + m + b + 1}{n - m}_r,$$

where the Euler coefficient [2] is denoted by

$$\binom{n + m + b + 1}{n - m}_r = \sum_{i=0}^{\lfloor (n-m)/r \rfloor} (-1)^i \binom{n + m + b + 1}{i} \binom{2n + b - ri}{n - m - ri}$$

(see Definition 4 and expansion (12)). More about Euler coefficients can be found in Section 4. For given m, the number of Dyck paths $Dyck\,(n + m, m - n; d^r)$ to $(n + m, m - n)$ avoiding d^r has the generating function (over n) $(1 - t^r)^m (1 - t)^{-m-2} (rt^r (1 - t) - (1 - t^r)(2t - 1))$, as shown in (11). Note that the coefficient of t^m in this generating function equals the original $Dyck\,(2m, 0; d^r)$.

Next we show that the number of Dyck paths to $(2n + b, 2m + b)$ avoiding u^r equals

$$\begin{aligned} & D\left(2m + b, 2m + b; u^4\right) \\ = & \sum_{i=0}^{2m+b-1} \frac{1}{n + m + b + 1 - i} \binom{i - 2m - b}{i}_r \binom{n + m + b + 1 - i}{n + m + b - i}_r, \end{aligned}$$

except for the original Dyck path counts to $(2n, 0)$, which either must be gotten from those to $(2n - 1, 1)$, or from the Dyck paths to $(2n, 0)$ avoiding

d^r. The case $r=4$ seems to be very special. We conjecture in Section 3.1 that in this case the generating function for the Dyck paths equals

$$\begin{aligned}&\sum_{n\geq 0} Dyck\left(4m-n-1, 2m-n+1; u^4\right)\\ = &\left(3+t-\sqrt{(1+t)^2+4t^3}\right)\left(\frac{1-t^4}{1-t}\right)^m /2,\end{aligned}$$

hence $Dyck\left(2m,0;u^4\right)=\left[t^{2m}\right]\left(3+t-\sqrt{(1+t)^2+4t^3}\right)\left(\frac{1-t^4}{1-t}\right)^m/2$.

Throughout the following sections we will discuss ballot paths (weakly above $y=x$), with steps $\uparrow$ and $\rightarrow$, instead of Dyck paths. The transformations $D(n,m)=Dyck(n+m,m-n)$ and $Dyck(2n+b,2m+b)=D(n-m,n+m+b)$, with $D(n,m)$ counting ballot path to (n,m), go back and forth between the two equivalent setups. Of course, the pattern u^r becomes the pattern $\uparrow^r$, or N^r, and d^r becomes $\rightarrow^r$, or E^r.

2 Ballot paths without the pattern $\rightarrow^r$

Definition 1 *$s_n(m;r)=s_n(m)$ is the number of $\{\uparrow,\rightarrow\}$ paths staying weakly above the diagonal $y=x$ from $(0,0)$ to $(n,m)\in\mathbb{Z}^2$ avoiding a sequence of $r>0$ consecutive $\rightarrow$ steps. We get $s_0(m)=1$ for all $m\geq 0$. We set $s_n(m)=0$ if $n<0$ or if $m+1=n>0$.*

Lemma 2 *The following recurrence holds for all $m\geq n>0$:*

$$s_n(m)=s_{n-1}(m)+s_n(m-1)-s_{n-r}(m-1). \tag{1}$$

Proof: The number of paths reaching (n,m) is obtained by adding the number of paths reaching $(n-1,m)$ and $(n,m-1)$, but subtracting paths that would have exactly $r\rightarrow$ steps. Those forbidden steps occur necessarily at the end of the path, so they are preceded by an up step, and must come from $(n-r,m-1)$. ■

We now extend $s_n(m)$ to all integers m by first setting $s_0(m)=1$ and using (1) to define the remaining $s_n(m)$ for $m<n-1$.

Lemma 3 *(s_n) is a polynomial sequence with* $\deg s_n=n$.

Proof: We proceed by induction on n. Clearly, $\deg(s_0)=0$. Suppose $s_k(m)$ is a polynomial of degree k for $0\leq k\leq l$. Then $s_{l+1}(m)-s_{l+1}(m-1)=s_l(m)-s_{l-r+1}(m-1)$, which implies the first difference of $s_{l+1}(m)$ is a polynomial of degree l. Thus, $s_{l+1}(m)$ is a polynomial of degree $l+1$. ■

By interpolation we can define (s_n) on all real numbers.

Definition 4 *The Eulerian Coefficient is defined as*

$$\binom{x}{n}_r = [t^n](1+t+\cdots+t^{r-1})^x = \sum_{i=0}^{\lfloor n/r \rfloor} (-1)^i \binom{x}{i}\binom{x+n-ri-1}{n-ri}$$

(see (12)). Note that for $r=2$ the Euler coefficient equals the binomial coefficient $\binom{x}{n}$.

The following table shows the polynomial extension of $s_n(m)$. The number of $\{\uparrow,\rightarrow\}$ paths to (n,m) avoiding a sequence of 4 $\rightarrow$ steps appear above the $y=x$ diagonal. The numbers on the diagonal (n,n), $1,1,2,5,13,36,\ldots$, are the number of Dyck paths to $(2n,0)$. Of course, $s_n(n) \leq C_n$, the n-th Catalan number.

m	1	7	27	75	161	273	357	309	0
6	1	6	20	48	87	118	104	0	-222
5	1	5	14	28	40	36	0	-76	-182
4	1	4	9	14	13	0	-27	-62	-93
3	1	3	5	5	0	-10	-22	-30	-31
2	1	2	2	0	-4	-8	-10	-8	-5
1	1	1	0	-2	-3	-3	-2	0	0
0	1	0	-1	-2	0	0	0	0	0
-1	1	-1	-1	-1	3	-1	-1	-1	3
$n:$	0	1	2	3	4	5	6	7	8

The path counts $s_n(m)$ and their polynomial extension ($r=4$)

Theorem 5

$$s_n(x) = \frac{x-n+1}{x+1}\binom{x+1}{n}_r = \frac{x-n+1}{x+1}\sum_{i=0}^{\lfloor n/r \rfloor}(-1)^i \binom{x+1}{i}\binom{x+n-ri}{n-ri}$$

Proof: We saw that $(s_n(x))$ is a basis for the vector space of polynomials. Using operators on polynomials, we can write the recurrence relation as

$$1 - E^{-1} = B - B^r E^{-1} \tag{2}$$

where B and E^a are defined by linear extension of $Bs_n(x) = s_{n-1}(x)$ and $E^a s_n(x) = s_n(x+a)$, the shift by a. The operators $\nabla = 1 - E^{-1}$ and E^{-1} both have power series expansions in D, the derivative operator. Hence

B must have such an expansion too, and therefore commutes with ∇ and E^a. The power series for B must be of order 1, because B reduces degrees by 1. Such linear operators are called *delta operators*. The basic sequence $(b_n(x))_{n\geq 0}$ of a delta operator B is a sequence of polynomials such that $\deg b_n = n$, $Bb_n(x) = b_{n-1}(x)$ (like the *Sheffer sequence* $s_n(x)$ for B), and initial conditions $b_n(0) = \delta_{0,n}$for all $n \in \mathbb{N}_0$. In our special case, the basic sequence is easily determined. Solving for E^1 in (2) shows that

$$E^1 = \sum_{i=0}^{r-1} B^i.$$

Finite Operator Calculus tells us that if $E^1 = 1 + \sigma(B)$, where $\sigma(t)$ is a power series of order 1 [3, (2.5)], then the basic sequence $b_n(x)$ of B has the generating function

$$\sum_{n\geq 0} b_n(x)t^n = (1+\sigma(t))^x.$$

Thus, in our case $b_n(x) = [t^n]\left(1 + t + t^2 + \cdots + t^{r-1}\right)^x = \binom{x}{n}_r$. Since the Sheffer sequence (s_n) has initial values $s_n(n-1) = \delta_{n,0}$, using Abelization [3] gives us

$$s_n(x) = \frac{x-n+1}{x+1} b_n(x+1) = \frac{x-n+1}{x+1}\binom{x+1}{n}_r. \tag{3}$$

■

Corollary 6 *The number of Dyck paths to $(2n,0)$ avoiding r down steps is*

$$s_n(n) = \frac{1}{n+1}\binom{n+1}{n}_r. \tag{4}$$

3 Ballot paths without the pattern $\uparrow^r$

Definition 7 $t_n(m;r) = t_n(m)$ *is the number of* $\{\uparrow, \rightarrow\}$ *paths staying weakly above the line* $y = x$ *from* $(0,0)$ *to* (n,m) *avoiding a sequence of* $r > 0$ *consecutive* $\uparrow$ *steps. We set* $t_n(m) = 0$ *if* $n < 0$ *or* $m+1 = n > 0$.

This time we do not immediately have a polynomial sequence, as the table below shows. The path $N^{r-1}\left(EN^{r-1}\right)^k$ to $(k,(r-1)(k+1))$ is the only admissible path reaching the point $(k,(r-1)(k+1))$ (all others would have r or more N-steps). Hence $t_{n-1}((r-1)n) = 1$ for all $n \geq 1$, and $t_{n-1}(m) = 0$ for $m > (r-1)n$. The only other 1's in the table occur in column 0, $t_0(m) = 1$ for $m = 0, \ldots, r-1$, and 0 for all other values of m.

The table contains a strip weakly above the diagonal $y = x$ where

$$t_n(m) = t_n(m-1) + t_{n-1}(m) \tag{5}$$

This happens for $0 < n \leq m < n + r$ because paths in this strip cannot have r consecutive vertical steps. All paths that reach a point (n, m) for $m \geq n + r$ and violate the condition of not containing N^r must have this pattern **exactly** at the end of the path, which means that they end in the pattern EN^r. Hence for $m \geq n + r$ we get the recurrence

$$t_n(m) = t_n(m-1) + t_{n-1}(m) - t_{n-1}(m-r) \tag{6}$$

We assume that $t_n(m) = 0$ for all $m < n$ (also for $n = 0$).

We can find a recursion that holds for all $m \geq n$ as follows: For $n \geq 1$ we always have $t_n(n) = t_{n-1}(n)$, because $t_n(n-1) = 0$. From (5) follows by induction (inside the exceptional strip) that $t_n(m) = \sum_{i=n}^{m} t_{n-1}(i)$ for all $n \leq m < n + r$. For the values of m on the boundary of the strip we have

$$t_n(n+r) = \sum_{i=n}^{n+r} t_{n-1}(i) - t_{n-1}(n) = \sum_{i=n+1}^{n+r} t_{n-1}(i)$$

from (5) and (6), and after that by induction using (6), $t_n(m) = \sum_{i=m+1-r}^{m} t_{n-1}(i)$ for all $m \geq n+r$. We can write both recursions together as

$$t_n(m) = \sum_{i=\max\{n, m+1-r\}}^{m} t_{n-1}(i) \tag{7}$$

for all $m \geq n$. We can avoid the difficulty with the lower bound in the summation by setting $t_n(m) = 0$ for all $m \leq n$. Call the modified numbers $t'_n(m)$.The new table follows the recursion

$$t'_n(m) = \sum_{i=m+1-r}^{m} t'_{n-1}(i)$$

for all $m > n > 0$. The 'lost' value $t_n(n)$ can be easily recovered, because $t_n(n) = t_{n-1}(n) = t'_{n-1}(n)$.

m									
m	0	0	1	19	112	397	1027	1966	2905
8	0	0	3	28	116	321	630	939	(939)
7	0	0	6	33	101	205	309	(309)	0
6	0	1	9	32	68	104	(104)	0	0
5	0	2	10	23	36	(36)	0	0	0
4	0	3	8	13	(13)	0	0	0	0
3	1	3	5	(5)	0	0	0	0	0
2	1	2	(2)	0	0	0	0	0	0
1	1	(1)	0	0	0	0	0	0	0
0	(1)	0	0	The values in parentheses are 0 in $t'_n(m)$					
$n:$	0	1	2	3	4	5	6	7	8

The restricted ballot path counts $t_n(m)$ $(r=4)$.

In order to show the polynomial structure in the above table, we transform it into the table below by a 90° counterclockwise turn, and shifting the top 1's flush against the y-axis. In formulas, we define $p_n(m) = t'_{m-1}((r-1)m-n)$ for $m(r-1) \geq n \geq 0$ (or $t'_n(m) = p_{(r-1)(n+1)-m}(n+1)$). The recursion $t'_n(m) = \sum_{i=m+1-r}^{m} t'_{n-1}(i)$ 'along the previous column' becomes now a recursion $p_n(m) = \sum_{i=0}^{r-1} p_{n-j}(m-1)$ 'along the previous row'. More precisely, for $(r-1)m - n > m-1 \geq 1$, i.e. , $m \geq 2$ and $n \leq (r-2)m$ holds

$$\begin{aligned} p_n(m) &= t'_{m-1}((r-1)m-n) = \sum_{i=(r-1)(m-1)-n}^{(r-1)m-n} t'_{m-2}(i) \\ &= \sum_{i=(r-1)(m-1)-n}^{(r-1)m-n} p_{(r-1)(m-1)-i}(m-1) = \sum_{i=0}^{r-1} p_{n-j}(m-1) \quad (8) \end{aligned}$$

The numbers $p_n(m)$ for $0 \leq n \leq (r-2)m$ are exactly the cases where $t'_n(m)$ is positive, and the only additional numbers needed in the recursion (8) are the numbers
$p_{(r-2)m-j}(m-1) = t'_{m-2}((r-1)(m-1) - (r-2)m + j) = t'_{m-2}(m + j - r + 1) = 0$ for $j = 0, \ldots, r-3$. We also add a row $p_n(0) = \delta_{n,0}$ for $n = 0, \ldots, r-2$ to the table, so that the recursion (8) holds for $m = 1$. This part of the p-table, $p_n(m)$ for $0 \leq m$ and $0 \leq n \leq (r-2)(m+1)$, is shown below for $r = 4$. Note that $p_0(m) = t_{m-1}((r-1)m) = 1$ for all $m \geq 1$, and also $p_0(0) = 1$.

m	1	8	36	119	315	699	1338	2246	3344
7	1	7	28	83	197	391	667	991	1295
6	1	6	21	55	115	200	297	379	419
5	1	5	15	34	61	90	112	116	**101**
4	1	4	10	19	28	33	**32**	23	13
3	1	3	6	9	**10**	8	5	0	0
2	1	2	**3**	3	2	0	0	*-2*	*2*
1	**1**	1	1	0	0	*-1*	*1*	*-2*	*4*
0	1	0	0	*-1*	*1*	*-1*	*2*	*-4*	*7*
$n:$	0	1	2	3	4	5	6	7	8

The rotated and shifted table $p_n(m)$ and its polynomial extension below the staircase, for $r = 4$. The bold numbers occur also on the second subdiagonal in the table below.

We obtained the recursion (7) as a discrete integral from (6) and (5). We can now take differences in recursion (8) and get $p_n(m) - p_{n-1}(m) = p_n(m-1) - p_{n-r}(m-1)$, or

$$p_n(m) - p_n(m-1) = p_{n-1}(m) - p_{n-r}(m-1)$$

for all $m \geq 1$ and $0 \leq n \leq (r-2)(m+1)$. The column $p_0(m)$ can be extended as a column of ones to all integers m; hence $p_0(m)$ can be extended to the constant polynomial 1. The recursion shows by induction that the n-th column can be extended to a polynomial of degree n, and by interpolation we can assume that we have polynomials in a real variable. The extension of $p_n(m)$ is again denoted by $p_n(m)$. The above table shows some values of the polynomial expansion in cursive. The expansion follows the same recursion, hence

$$p_n(x) - p_n(x-1) = p_{n-1}(x) - p_{n-r}(x-1) \tag{9}$$

with initial values $p_{(r-2)m+j}(m) = 0$ for $j = 1, \ldots, r-2$ and $m \geq 0$. These conditions, together with $p_0(0) = 1$, determine the solution uniquely.

Recursion (9) shows that $(p_n(x))$ is a Sheffer sequence for the same operator B as the sequences $(s_n(x))$ in recursion (2). Hence $p_n(x)$ can be written in terms of the same basis, the Eulerian coefficients, as $s_n(x)$. However, the initial values (zeroes) for $(p_n(x))$ are more difficult, because they are not on a line with positive slope. We introduce know a Sheffer sequence $(q_n(x;\alpha))_{n \geq 0}$ for the delta operator B that has roots on the parallel to the diagonal shifted by $\alpha + 1$, $q_n(n - \alpha - 1; \alpha) = 0$, and agrees with (p_n) at one position left of the roots, for each n.

Lemma 8 *For the Sheffer sequence* $(q_n(x;\alpha))$ *for* B *with initial values* $q_0(m;\alpha)=1$, $q_n(0)=\delta_{n,0}$ *for* $0\leq n\leq\alpha$ *and* $q_n(n-\alpha-1;\alpha)=0$ *for* $n>\alpha$,
holds

$$q_{n+\alpha}(n;\alpha)=p_{(r-2)n-\alpha}(n)$$

for all $n\geq\lceil\alpha/(r-2)\rceil$.

We will proof this Lemma in Subsection 3.2.

m									
m	1	6	21	56	120	214	320	386	**321**
5	1	5	15	35	65	99	121	**101**	0
4	1	4	10	20	31	38	**32**	0	-70
3	1	3	6	10	12	**10**	0	-22	-58
2	1	2	3	4	**3**	0	-7	-18	-33
1	1	1	1	**1**	0	-2	-6	-10	-15
0	1	0	0	0	0	-2	-4	-4	-5
$n:$	0	1	2	3	4	5	6	7	8

The polynomials $q_n(m,2)$ for $r=4$

The sequence (q_n) agrees with the Euler coefficients $b_n(x)=\binom{x}{n}_r$ for the first degrees $n=0,\ldots,\alpha$. It follows from the Binomial Theorem for Sheffer sequences that

$$q_n(x;\alpha)=\sum_{i=0}^{\alpha}\binom{i-\alpha-1}{i}_r\frac{x+\alpha+1-n}{x+\alpha+1-i}\binom{x+\alpha+1-i}{n-i}_r. \tag{10}$$

Corollary 9 *The number of ballot paths avoiding* r $\uparrow$*-steps equals*

$$t_n(m;r)=\sum_{i=0}^{m-n-1}\frac{1}{m+1-i}\binom{i-m+n}{i}_r\binom{m+1-i}{m-i}_r$$

for $m>n\geq 0$. *Furthermore,* $t_n(n)=t_{n-1}(n)=\frac{1}{n+1}\binom{n+1}{n}_r$ *for all* $n>0$.

Proof: $t_n(m;r)=p_{(r-1)(n+1)-m}(n+1)=q_m(n+1;m-1-n)$. ∎

The Corollary shows that the number of Dyck paths to $(2n,0)$ avoiding r up steps, $\frac{1}{n+1}\binom{n+1}{n}_r$, equals the number of Dyck paths to $(2n,0)$ avoiding r down steps (see formula (4)).

3.1 A Conjecture for the Case $r = 4$.

A Motzkin path can take horizontal unit steps in addition to the up and down steps of a Dyck path. Suppose a Motzkin path is "peakless", i.e., the pattern uu and ud does not occur in the path. Denote the number of peakless Motzkin paths to $(n,0)$ by $M'(n)$. Starting at $n = 0$ we get the following sequence, $1,1,1,2,4,7,13,26,52,104,212,438,910,\ldots$ for $M'(n)$ (see http://www.research.att.com/~njas/sequences/A023431). It is easy to show that $M'(n) = \sum_{i=0}^{\lfloor n/3\rfloor} \binom{n-i}{2i} \frac{1}{i+1} \binom{2i}{i}$.

For the case $r = 4$ we conjecture that $p_n(0) = (-1)^n M'(n-3)$ for all $n \geq 3$. That would imply $\sum_{n\geq 0} p_n(0)\, t^n = \left(3 + t - \sqrt{(1+t)^2 + 4t^3}\right)/2$, and therefore

$$\sum_{n\geq 0} p_n(x)\, t^n = \frac{3 + t - \sqrt{(1+t)^2 + 4t^3}}{2} \left(\frac{1-t^4}{1-t}\right)^x .$$

For example, the coefficient of t^7 in this power series equals $(x-3)\left(x^6 + 24x^5 + 247x^4 + 426x^3 - 38x^2 - 2340x + 6720\right)/7!$, which in turn equals $p_7(x)$, as can be checked using the table for $p_7(m)$.

3.2 Proof of Lemma 8

Because of recursion (8) we obtain the operator identity

$$I = E^{-1}\left(B^0 + B^1 + \cdots + B^{r-1}\right)$$

which holds for $(p_n(x))$ and $(q_n(x;\alpha))$, and shows that both polynomials enumerate lattice paths with steps $\langle 0,1\rangle, \langle 1,1\rangle, \langle 2,1\rangle, \ldots, \langle r-1,1\rangle$ (above the respective boundaries). The number of such paths reaching (n,m) can also be seen as compositions of m into n terms taken from $\{0,1,\ldots,r-1\}$. In the case of $p_n(m)$ the terms $a_1,\ldots,a_n$ also have to respect the boundary, which means that $\sum_{i=1}^{k} a_i \leq (r-2)\,k$ for all $k = 1,\ldots n$. For $q_n(m;\alpha)$ we get for the same reason that $\sum_{i=1}^{k} b_i \leq k + \alpha - 1$. Such restricted compositions have the following nice property.

Lemma 10 *Let $c \in \mathbb{N}_1$, $\alpha \in \mathbb{N}_0$, and n be a natural number such that $n - \alpha \geq 0$. Let P_n^α be the number of compositions of $cn - \alpha$ into n parts from $[0, c+1]$ such that $a_1 + a_2 + \ldots + a_n = cn - \alpha$, and $\sum_{i=1}^{k} a_i \leq ck$ for all $k = 1,\ldots,n-1$. Let Q_n^α be the number of compositions of $n + \alpha$ into n*

parts from $[0, c+1]$ *such that* $b_1 + b_2 + \ldots + b_n = n + \alpha$*, and* $\sum_{i=1}^{k} b_i \leq k + \alpha$ *for all* $k = 1, \ldots, n-1$*. Then* $P_n^\alpha = Q_n^\alpha$*.*

Proof: Suppose, $b_1 + b_2 + \ldots + b_n = n + \alpha$, and $\sum_{i=1}^{k} b_i \leq k + \alpha$. Define $a_i = c+1-b_{n+1-i}$ for $i = 1, \ldots, n$. Note that $a_i \in [0, c+1]$, and $n-k+\alpha \geq \sum_{i=1}^{n-k} b_i = n + \alpha - \sum_{i=1}^{k} b_{n+1-i}$. Hence

$$\sum_{i=1}^{k} a_i = (c+1)k - \sum_{i=1}^{k} b_{n+1-i} \leq (c+1)k - n + (n-k) = ck$$

and

$$\sum_{i=1}^{n} a_i = (c+1)n - \sum_{i=1}^{n} b_{n+1-i} = cn - \alpha.$$

■

We apply this Lemma with $c = r-2$ to obtain $p_{(r-2)n-\alpha}(n) = q_{n+\alpha}(n)$.

3.3 Abelization

Let (b_n) be the basic sequence for some arbitrary delta operator B, i.e., $Bb_n = b_{n-1}$ and $b_n(0) = \delta_{0,n}$. Every basic sequence is also a sequence of binomial type, which means that $\sum_{n \geq 0} b_n(x) t^n = e^{x\beta(t)}$, where $\beta(t) = t + a_2 t^2 + \ldots$ is a formal power series. The compositional inverse of $\beta(t)$ is the power series that represents B,

$$B = \beta^{-1}(D) = D + b_2 D^2 + \ldots$$

where $D = \partial/\partial x$ is the x-derivative. The Abelization of (b_n) (by $a \in \mathbb{R}$) is the basic sequence $\left(\frac{x}{x+an} b_n(x+an)\right)_{n \geq 0}$ for the delta operator $E^{-a}B$ (see [4]). Note that with any Sheffer sequence (s_n) for B the sequence $(s_n(x+c-an))$ is a Sheffer sequence for $E^a B$. Hence $\left(\left(\frac{x+c-an}{x+c} b_n(x+c)\right)\right)_{n \geq 0}$ is a Sheffer sequence for $E^a E^{-a} B = B$ again. Choosing $c = a = 1$ shows (3).

Sheffer sequences and the basic sequence for the same delta operator are connected by the Binomial Theorem for Sheffer sequences,

$$s_n(y+x) = \sum_{i=0}^{n} s_i(y) b_{n-i}(x).$$

Applying this Theorem to $\left(\frac{x}{x+an} b_n(x+an)\right)$ and $(b_n(x+c+an))$ shows that

$$b_n(y+x+c+an) = \sum_{i=0}^{n} b_i(y+c+ai) \frac{x}{x+a(n-i)} b_{n-i}(x+a(n-i)).$$

Choosing x as $x+\alpha+1-an$, $a=1$, $c=0$, and $y=-\alpha-1$ gives $b_n(x) = \sum_{i=0}^{n} b_i(i-\alpha-1) \frac{x+\alpha+1-n}{x+\alpha+1-i} b_{n-i}(x+\alpha+1-i)$.This is not quite what we have in (10); there the summation stops at α. This effect in (10) is due to the 'initial values' $b_i(i-\alpha-1)$ which are 0 for $i>\alpha$.

The generating function of a Sheffer sequence (s_n) for B is of the form $\phi(t) e^{x\beta(t)}$, where $\phi(t) = \sum_{n\geq 0} s_n(0) t^n$. If $s_n(x) = \frac{x+c-an}{x+c} b_n(x+c) = b_n(x+c) - \frac{an}{x+c} b_n(x+c)$ then

$$\sum_{n\geq 1} \frac{n}{x+c} b_n(x+c) t^n = \frac{t}{x+c} \frac{\partial}{\partial t} e^{(x+c)\beta(t)} = t\beta'(t) e^{(x+c)\beta(t)}$$

and

$$\sum_{n\geq 0} s_n(x) t^n = e^{(x+c)\beta(t)} \left(1 - at\beta'(t)\right)$$

If $c=a=1$ and $e^{\beta(t)} = \left(1+t+\cdots+t^{r-1}\right)$, then $\beta'(t) = \frac{1-(r-tr+t)t^{r-1}}{(1-t^r)(1-t)}$, and therefore

$$\sum_{n\geq 0} s_n(m) t^n = \frac{(1-t^r)^m}{(1-t)^{m+2}} \left(rt^r(1-t) + (1-t^r)(1-2t)\right), \tag{11}$$

the generating function of the number of ballot paths to (n,m), avoiding $\rightarrow^r$.

4 Euler Coefficients

The coefficients of the polynomial

$$\left(1+t+t^2+\cdots+t^{r-1}\right)^n$$

were considered by Euler in [2], where he gives the following recurrence:

$$\binom{n}{k}_{r+1} = \sum_{i=0}^{k/2} \binom{n}{k-i} \binom{k-i}{i}_r$$

To calculate the Euler coefficients in terms of only binomial coefficients, we rewrite the polynomial as follows:

$$\begin{aligned}\left(1+t+t^2+\cdots+t^{r-1}\right)^n &= \left(\frac{1-t^r}{1-t}\right)^n \\ &= \sum_{i=0}^{n}\binom{n}{i}t^{ri}(-1)^i\sum_{j\geq 0}\binom{n+j-1}{j}t^j.\end{aligned}$$

Thus we have proven

$$\binom{n}{k}_r = \sum_{i=0}^{\lfloor k/r\rfloor}(-1)^i\binom{n}{i}\binom{n+k-ri-1}{k-ri}. \tag{12}$$

Note that this identity implies $\lim_{r\to\infty}\binom{n}{k}_r = \binom{n+k-1}{k}$. Combinatorially, these are *all* $\{\uparrow,\rightarrow\}$ paths avoiding $\rightarrow^r$. This problem occurs in Wilf's *generatingfunctionology* [6], Section 4.12. Note that identity (12) implies $\lim_{r\to\infty}\binom{n}{k}_r = \binom{n+k-1}{k}$.

x									
x	1	8	36	120	322	728	1428	2472	3823
7	1	7	28	84	203	413	728	1128	1554
6	1	6	21	56	120	216	336	456	546
5	1	5	15	35	65	101	135	155	155
4	1	4	10	20	31	40	31	20	10
3	1	3	6	10	12	12	10	6	3
2	1	2	3	4	3	2	1	0	0
1	1	1	1	1	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
n :	0	1	2	3	4	5	6	7	8

A table of Euler coefficients $\binom{x}{n}_4$ for $r=4$

We now show some properties about Euler Coefficients similar to the basic properties of binomial coefficients.

1. For binomial coefficients, this property is usually called Pascal's Identity:

$$\binom{n}{k}_r = \sum_{i=0}^{r-1}\binom{n-1}{k-i}_r$$

Proof

$$\begin{aligned}(1+t+\cdots+t^{r-1})^n &= (1+t+\cdots+t^{r-1})^{n-1}(1+t+\cdots+t^{r-1})\\ &= \sum_{i=0}^{r-1} t^i(1+t+\cdots+t^{r-1})^{n-1}\end{aligned}$$

so

$$\binom{n}{k}_r = \sum_{i=0}^{r-1}[t^{k-i}](1+t+\cdots+t^{r-1})^{n-1} = \sum_{i=0}^{r-1}\binom{n-1}{k-i}_r$$

■

2. The table of Euler Coefficients is symmetric similar to Pascal's Triangle:
$$\binom{n}{k}_r = \binom{n}{n(r-1)-k}_r$$
Proof We proceed by induction on n, fixing $r \geq 2$. For $n = 2$ we have the well known symmetry for binomial coefficients. Suppose true for some $l > 2$. From the above recurrence we have
$$\begin{aligned}\binom{l+1}{k}_r &= \sum_{i=0}^{r-1}\binom{l}{k-i}_r = \sum_{i=0}^{r-1}\binom{l}{l(r-1)-(k-i)}_r\\ &= \sum_{i=0}^{r-1}\binom{l}{(l+1)(r-1)-k-i}_r = \binom{l+1}{(l+1)(r-1)-k}_r\end{aligned}$$
and the induction follows. ■

3. This property is similar to Vandermondt Convolution for binomial coefficients:
$$\binom{n+m}{k}_r = \sum_{i=0}^{k}\binom{n}{i}_r\binom{m}{k-i}_r.$$
It follows because the Euler coefficients are of binomial type [4].

4. Here we have an identity that is trivial for binomial coefficients, i.e. $r = 2$, and gives and identity for the Catalan numbers as $r \to \infty$.
$$\frac{1}{n+1}\binom{n+1}{n}_r = \binom{n}{n}_r - \sum_{i=1}^{r-2} i\binom{n}{n-i-1}_r$$
Proof Let $s_n(x) = \dfrac{x-n+1}{x+1}\dbinom{x+1}{n}_r$, as in (3). The binomial theorem for Sheffer sequences states that $s_n(x+y) = \sum_{i=0}^{n} s_i(y)b_{n-i}(x)$.

Let $x = n$, $y = 0$ and noting that $s_n(0) = (1-n)\binom{1}{n}_r = 1 - n$ for $0 < n < r$ and 0 otherwise, we have $s_n(n) = \sum_{i=0}^{n} s_i(0) b_{n-i}(n)$, hence

$$\begin{aligned} \frac{1}{n+1}\binom{n+1}{n}_r &= \sum_{i=0}^{r-1}(1-i)\binom{n}{n-i}_r \\ &= \binom{n}{n}_r - \sum_{i=1}^{r-2} i\binom{n}{n-i-1}_r. \end{aligned}$$

We have already noted that $\binom{n}{k}_r \to \binom{n+k-1}{k}$ as $r \to \infty$, so

$$\lim_{r\to\infty} \frac{1}{n+1}\binom{n+1}{n}_r = \binom{2n-1}{n} - \sum_{i=1}^{n-1} i\binom{2n-i-2}{n-1} = C_n.$$

■

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